TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 296, Number 1, July 1986

THE SPECTRUM $(P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty}$

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ABSTRACT. The spectrum $(P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty}$ is defined to be the homotopy inverse limit of spectra $P_{-k} \wedge \mathrm{BP}\langle 2\rangle$, where P_{-k} is closely related to stunted real projective spaces, and $\mathrm{BP}\langle 2\rangle$ is formed from the Brown-Peterson spectrum. It is proved that this spectrum is equivalent to the infinite product of odd suspensions of the 2-adic completion of the spectrum of connective K-theory. An odd-primary analogue is also proved.

1. Introduction. In [**12** and **8**] an inverse system

$$(1.1) \qquad \cdots \to P_{-k-1} \to P_{-k} \to \cdots \to P_0$$

of spectra constructed from stunted real projective spaces was considered. If E is any spectrum, the homotopy inverse limit of the system obtained by applying $\wedge E$ to (1.1) is denoted by $(P \wedge E)_{-\infty}$.

Let BP denote the Brown-Peterson spectrum associated to the prime p [**6**], and $\mathrm{BP}\langle n\rangle$ the associated spectrum constructed in [**4**] and studied in [**9**] which satisfies

$$\pi_*(\mathrm{BP}\langle n\rangle) = \mathbf{Z}_{(p)}[v_1, \dots, v_n].$$

A corrected conjecture of [**8**] is that there is an equivalence of spectra

$$(1.2) \qquad (P \wedge \mathrm{BP}\langle n\rangle)_{-\infty} \approx \prod_{k \in \mathbf{Z}} \Sigma^{2k-1} \hat{\mathrm{BP}}\langle n-1\rangle,$$

where the BP's are associated to $p = 2$, and $\hat{E}$ denotes the 2-adic completion of the spectrum E. This was proved when $n = 1$ in [**8**]. The purpose of this paper is to prove the cases $n = 2$ and $n = \infty$ of Conjecture 1.2 and a generalization to every prime.

Before we embark upon the constructions required to state the theorem, especially in the case when p is odd, we sketch the intuition behind the result and proof when $p = 2$. We have

$$\pi_*((P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty}) \approx \operatorname{inv\,lim} \pi_*(P_{-2k-1} \wedge \mathrm{BP}\langle 2\rangle),$$

Received by the editors November 26, 1984 and, in revised form, May 3, 1985 and July 23, 1985.
1980 *Mathematics Subject Classification*. Primary 55P42; Secondary 55N22, 55T15.
Key words and phrases. Brown-Peterson spectrum, homotopy inverse limits, projective spaces, Adams spectral sequence.
[1] Supported by NSF grants.
[2] Supported by an NSERC postgraduate fellowship and an Imperial Oil graduate fellowship.

which can be calculated by standard methods (see e.g. (4.2)) to be $\prod_{k \in \mathbf{Z}} \pi_*(\Sigma^{2k-1}bu)$. In order to topologically realize this isomorphism, we must construct compatible maps

$$\Sigma^{2i}bu \to \Sigma P_{2k-1} \wedge \mathrm{BP}\langle 2\rangle.$$

Using cofibrations related to the fact that the sphere bundle of $H \otimes H$ over $\mathbf{C}P^n$ is P^{2n+1}, we construct a map from a spectrum C_k into $\Sigma P_{2k-1} \wedge \mathrm{BP}\langle 2\rangle$, where

$$H^*(C_k) \approx H^*\Big(\bigvee_{i \geqslant k} \Sigma^{2i}bu\Big).$$

A novel spectral sequence argument (§5) shows that such a cohomology isomorphism can be realized as an equivalence of spectra. The maps $\Sigma^{2i}bu \to \Sigma P_{2k-1} \wedge \mathrm{BP}\langle 2\rangle$ will not necessarily be compatible as k decreases, but their existence is used in another spectral sequence argument (§4) to establish the existence of the desired maps

$$\Sigma^{2i}bu \to \Sigma(P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty}.$$

We review the following constructions of [**15** and **7**]. Let p be any prime and $q = 2(p-1)$. There is a complex $(p-1)$-plane bundle β over $B\Sigma_p$, which, when restricted to $B\mathbf{Z}/p$, has sphere bundle equivalent to that of $(p-1)\lambda$, where λ is the canonical line bundle. Thus for any integer k there is a map

$$T(k(p-1)\lambda) \to T(k\beta), \tag{1.3}$$

where $T(\ )$ denotes the Thom spectrum. We denote the spectra in (1.3) by L_{qk} and P_{qk}, respectively. Note that L_{qk} has one cell of each dimension $\geqslant qk$, while P_{qk} has one cell of each dimension $\geqslant qk$ which is congruent to 0 or $-1 \bmod q$. By [**7**, 1.1] appropriate skeleta of L_{qk} and P_{qk} are stably equivalent to stunted lens spaces and stunted $B\Sigma_p$'s, respectively. Thus there are compatible collapse maps c

$$\begin{array}{ccc} L_{qk} & \xrightarrow{c} & L_{q(k+1)} \\ \downarrow & & \downarrow \\ P_{qk} & \xrightarrow{c} & P_{q(k+1)}, \end{array}$$

and by collapsing intermediate cells we can define $L_n = L_{qk}/L_{qk}^{(n-1)}$ if $qk < n$, and $P_{q(k+1)-1} = P_{qk}/S^{qk}$, and obtain inverse systems

$$\begin{array}{ccccccccccc} \cdots \to & L_{-q(k+1)} & \to \cdots \to & L_{-qk-1} & \to & L_{-qk} & \to \cdots \to & L_0 \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \to & P_{-q(k+1)} & \to & P_{-qk-1} & \to & P_{-qk} & \to \cdots \to & P_0. \end{array}$$

Note that if $p = 2$, then $L_n = P_n$ for all n, and they agree with the spectra of (1.1).

DEFINITION 1.4. If E is a p-local spectrum, then $(P \wedge E)_{-\infty}$ is the homotopy inverse limit of

$$\cdots \to P_{-q(k+1)} \wedge E \to P_{-qk-1} \wedge E \to \cdots \to P_0 \wedge E,$$

and $(L \wedge E)_{-\infty}$ is the homotopy inverse limit [**5**] of

$$\cdots \to L_{-n-1} \wedge E \to L_{-n} \wedge E \to \cdots \to L_0 \wedge E.$$

THEOREM 1.5. *If p is any prime and $q = 2(p-1)$, there are equivalences of p-complete spectra*

$$(P \wedge \mathrm{BP}\langle 2\rangle)_{-\infty} \approx \prod_{k\in\mathbf{Z}} \Sigma^{qk-1}\hat{\mathrm{BP}}\langle 1\rangle,$$
$$(L \wedge \mathrm{BP}\langle 2\rangle)_{-\infty} \approx \prod_{k\in\mathbf{Z}} \Sigma^{2k-1}\hat{\mathrm{BP}}\langle 1\rangle,$$

where $\hat{E}$ denotes the p-adic completion of the spectrum E.

The proof of 1.5 utilizes the splitting [**11**, **13**] of $\mathrm{BP}\langle 1\rangle \wedge \mathrm{BP}\langle 1\rangle$, for which a $\mathrm{BP}\langle n\rangle$-analog for $n > 1$ has not been established. It also utilizes Robinson's Kunneth theorem for connective K-theory [**17**], for which a $\mathrm{BP}\langle n\rangle$-analog is not apparent. Thus, it will not be easy to generalize 1.5 to (1.2). However, it is not difficult to prove the $n = \infty$ version below.

THEOREM 1.6. *There are equivalences of p-local spectra*

$$(P \wedge \mathrm{BP})_{-\infty} \approx \prod_{k\in\mathbf{Z}} \Sigma^{qk-1}\hat{\mathrm{BP}}, \quad (L \wedge \mathrm{BP})_{-\infty} \approx \prod_{k\in\mathbf{Z}} \Sigma^{2k-1}\hat{\mathrm{BP}}.$$

Theorem 1.5 follows from Theorem 2.3, which is proved in §4. The proof of 2.3 utilizes 2.1 and 2.2, which are proved in §3. The proof of Theorem 1.6 appears in §6.

We thank John Jones, Wolfgang Lellmann, and Doug Ravenel for valuable conversations.

2. Sketch of proof of Theorem 1.5. Many Adams spectral sequence (ASS) arguments are involved in the proof of 1.5; however, a sketch of the argument can be presented without resorting to these.

First we establish some notation. We let p be any prime and $q = 2(p-1)$. The spectra $\mathrm{BP}\langle 2\rangle$ and $\mathrm{BP}\langle 1\rangle$ will be abbreviated to B and l, respectively. The latter is consistent with [**11** and **7**]. If $p = 2$, then $l = bu$, the spectrum of connective complex K-theory localized at 2, while for odd primes l is a summand in Adams' splitting of bu localized at p.

A denotes the mod p Steenrod algebra, E_2 the exterior subalgebra generated by Milnor primitives Q_0, Q_1, and Q_2, with $|Q_i| = 2p^i - 1$, and E_1 the subalgebra generated by Q_0 and Q_1. Then $H^*(l) \approx A//E_1 = A \otimes_{E_1} F_p$ and $H^*(B) \approx A//E_2$, where F_p denotes $\mathbf{Z}/p\mathbf{Z}$ and all cohomology groups have F_p-coefficients. Also, recall that, for any integer k, $H^*(L_k)$ is the submodule of classes of degree $\geqslant k$ in $\Delta = E[x] \otimes F_p[y^{\pm 1}]$, where $|x| = 1$, $|y| = 2$, $\beta x = -y$, $\mathcal{P}^a x = 0$ if $a > 0$, and $\mathcal{P}^a y^b = \binom{b}{a} y^{b+a(p-1)}$. Here $\mathcal{P}^a$ denotes the Steenrod operation ($= \mathrm{Sq}^{2a}$ if $p = 2$), and $\beta = Q_0$ is the Bockstein.

$B_* = \pi_*(B) \approx \mathbf{Z}_{(p)}[v_1, v_2]$ with $|v_i| = 2(p^i - 1)$. The p-series $[p](X)$ is a power series with coefficients in B_*. It begins

$$pX - (p^{p-1} - 1)v_1 X^p + p^{p-1}(p^{p-1} - 1)v_1^2 X^{2p-1} + \cdots.$$

Theorem 1.5 follows from the next three results, the first two of which are proved in §3 and the last in §4.

Spectra $\mathbf{C}P_k$ for any integer k can be constructed from stunted complex projective spaces similarly to the real analogs P_k, either as Thom spectra or using James periodicity.

THEOREM 2.1. *For all integers k there are spectra T_k and cofibrations*

$$L_{2k-1} \to \mathbf{C}P_k \xrightarrow{q} T_k \to \Sigma L_{2k-1}.$$

$B_*(T_k)$ and $B_*(\mathbf{C}P_k)$ are free B_*-modules on generators $\gamma_i \in B_{2i}(T_k)$ and $\beta_i \in B_{2i}(\mathbf{C}P_k)$, respectively, $i \geqslant k$. Moreover, if $[p](X) = \sum c_j X^{1+(p-1)j}$, then $q_*(\beta_i) = \sum c_j \gamma_{i-(p-1)j}$.

THEOREM 2.2. *For each integer k, there is a map of cofibrations*

$$\begin{array}{ccccc} \mathbf{C}P_k \wedge B & \xrightarrow{q\wedge 1} & T_k \wedge B & \to & \Sigma L_{2k-1} \wedge B \\ \uparrow & & \uparrow & & \uparrow g \\ \bigvee_{i \geqslant k+p^2-1} \Sigma^{2i} B & \xrightarrow{\tilde{q}} & \bigvee_{i \geqslant k} \Sigma^{2i} B & \to & C_k \end{array}$$

with $C_k \simeq \bigvee_{i \geqslant k} \Sigma^{2i} l$. In $H^*(\ )$, $g^*(\Sigma x y^{i-1} \otimes 1) = \Sigma(-1)^j \mathscr{P}^j G_{2i-qj}$, where G_{2i} generates $H^*(\Sigma^{2i} l) \subset H^*(C_k)$.

THEOREM 2.3. (i) *For any integer i, there is a map $\Sigma^{2i-1} l \to (L \wedge B)_{-\infty}$, such that, for any integer k, the cohomology homomorphism induced by the composite*

$$\Sigma^{2i-1} l \to (L \wedge B)_{-\infty} \to L_{2k-1} \wedge B$$

sends $xy^{i-1+(p-1)j} \otimes 1$ to $(-1)^j \mathscr{P}^j G_{2i-1}$.

(ii) *The maps of* (i) *induce an equivalence $(\bigvee_{i \in \mathbf{Z}} \Sigma^{2i-1} l)^\wedge \to (L \wedge B)_{-\infty}$.*

(iii) *There is an equivalence $(\bigvee_{i \in \mathbf{Z}} \Sigma^{2i-1} l)^\wedge \to \prod_{k \in \mathbf{Z}} \Sigma^{2k-1} \hat{l}$.*

(iv) *There are equivalences $\prod_k \Sigma^{qk-1} \hat{l} \leftarrow (\bigvee_{i \in \mathbf{Z}} \Sigma^{qi-1} l)^\wedge \to (P \wedge B)_{-\infty}$.*

3. Proof of Theorems 2.1 and 2.2.

PROOF OF 2.1. Let T denote the Thom spectrum of the complex line bundle $\otimes^p H$ over $\mathbf{C}P^\infty$, and T^k its $2k$-skeleton. There is a cofibration

$$L^{2k-1} \xrightarrow{h} \mathbf{C}P^{k-1} \xrightarrow{q} T^k \xrightarrow{a} \Sigma L^{2k-1},$$

where h is the canonical map, and L^{2k-1} denotes the skeleton of the lens space BF_p. If q is made skeletal, then the mapping cone $\mathrm{MC}(\mathbf{C}P^{k-1} \xrightarrow{q} T^{k-1})$ is ΣL^{2k-2}, so that a commutative diagram of cofibrations

$$\begin{array}{ccccccc} L^{2k-2} & \xrightarrow{h} & \mathbf{C}P^{k-1} & \xrightarrow{q} & T^{k-1} & \to & \Sigma L^{2k-2} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L^\infty & \xrightarrow{h} & \mathbf{C}P^\infty & \xrightarrow{q} & T & \to & \Sigma L^\infty \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ L_{2k-1} & \xrightarrow{h} & \mathbf{C}P_k & \xrightarrow{q} & T_k & \to & \Sigma L_{2k-1} \end{array}$$

is obtained, defining T_k when $k > 0$. That $B_*(q)$ corresponds to the p-series is well known (see e.g. [**19**]); this is one definition of the p-series.

Let A_n denote the J-order of the Hopf bundle over $\mathbf{C}P^n$, the Atiyah-Todd number [**3**]. If $N \equiv 0\ (A_{k-2})$, then there are J-trivializations of the bundles $2N\xi_{2k-3}$ and nH_{k-2} compatible with the natural map between them [**3, 10**]. Thus the identification of stunted lens spaces as Thom complexes shows that if $N \equiv 0\ (A_{k-2})$, then there is a commutative diagram

$$\begin{array}{ccc} \Sigma^{2N} L_1^{2k-2} & \xrightarrow{h} & \Sigma^{2N} \mathbf{C}P_1^{k-1} \\ \downarrow \simeq & & \downarrow \simeq \\ L_{2N+1}^{2N+2k-2} & \xrightarrow{h} & \mathbf{C}P_{N+1}^{N+k-1} \end{array}$$

and these can be chosen compatibly as N and k are varied. Thus, for k negative, T_k can be defined as the spectrum whose $(2k + 2M)$-skeleton T_k^{k+M} satisfies

$$\Sigma^{2N} T_k^{k+M} = \mathrm{MC}\left(L_{2k-1+2N}^{2k+2M+2N} \xrightarrow{h} \mathbf{C}P_{k+N}^{k+M+N} \right)$$

if $N \equiv 0\ (A_M)$ and $k + N > 0$. That this is well defined and satisfies 2.1 follows from the above remarks. □

Now we work toward the proof of 2.2. We form the diagram

$$(3.1) \qquad \begin{array}{ccc} \mathbf{C}P_k \wedge B & \xrightarrow{q \wedge 1} & T_k \wedge B \\ h_1 \uparrow \simeq & & h_2 \uparrow \simeq \\ \bigvee_{i \geqslant k} \Sigma^{2i} B & \xrightarrow{q'} & \bigvee_{i \geqslant k} \Sigma^{2i} B \end{array}$$

where h_1 and h_2 are constructed by applying $m \circ (\ \wedge B)$ to maps $S^{2i} \to \mathbf{C}P_k \wedge B$ (resp. $T_k \wedge B$) representing the generators of Theorem 2.1. Here m is the multiplication of the ring spectrum B [**16**]. The map q' which makes the diagram commute is constructed similarly from $\Sigma c_j \gamma_{i-(p-1)j}$. Let $\tilde{q}$ denote the restriction of q' to all but the first $p^2 - 1$ summands, and C_k the cofiber of $\tilde{q}$. The map g in 2.2 is the induced map of cofibers and is a B-module map.

Next we prove

PROPOSITION 3.2. *$H^*(C_k) \approx \oplus_{i \geqslant k} \Sigma^{2i} A//E_1$ as A-modules with generators G_{2i} satisfying the equation of* 2.2.

PROOF. By the uniformity of the homomorphisms q_*, it suffices to consider $k = 0$. Since h_2 in (3.1) is an equivalence, there is a commutative diagram of cofibrations

$$\begin{array}{ccccc} C_0 & \xrightarrow{g} & \Sigma L_{-1} \wedge B & \xrightarrow{k} & \bigvee_{i=0}^{p^2-2} \Sigma^{2i+1} B \\ \downarrow & & \downarrow & & \downarrow = \\ \bigvee_{i \geqslant p^2-1} \Sigma^{2i+1} B & \to & \bigvee_{i \geqslant 0} \Sigma^{2i+1} B & \to & \bigvee_{i=0}^{p^2-2} \Sigma^{2i+1} B. \end{array}$$

Since $k^*(\Sigma^{2i+1} l) = \Sigma y^i \otimes 1$,

$$H^*(C_0) \approx \left(\alpha_0, \alpha_2, \dots : Q_0 \alpha_{2i} = Q_1 \alpha_{2i-q} = Q_2 \alpha_{2i-2p^2+2} \right),$$

where $\alpha_{2i} = g^*(\Sigma xy^{i-1} \otimes 1)$, and if $n < 0$ then α_n is interpreted to be 0. Here we use that the A-module $H^*(\Sigma L_{-1} \wedge B)$ is generated by $\{\Sigma xy^{j-1} \otimes 1\colon\ j \geqslant 0\}$ with relations

$$Q_0(\Sigma xy^{j-1} \otimes 1) = Q_1(\Sigma xy^{j-p} \otimes 1) = Q_2(\Sigma xy^{j-p^2} \otimes 1),$$

where terms are ignored (not set $= 0$) if the exponent of y is less than -1.

An isomorphism Φ of the above presentation of $H^*(C_0)$ with $\oplus \Sigma^{2i}A//E_1$ is given by

$$\Phi(\alpha_{2i}) = \sum_{j=0}^{[i/p-1]} (-1)^j \mathscr{P}^j G_{2i-qj}.$$

To verify that Φ is well defined, one uses

(3.3) $$\mathscr{P}^j Q_i - Q_i \mathscr{P}^j = Q_{i+1}\mathscr{P}^{j-p^i}$$

to show $\Phi(Q_0\alpha_{2i}) = \Phi(Q_1\alpha_{2i-q}) = \Phi(Q_2\alpha_{2i-2p^2+2})$. That Φ is an isomorphism follows by a counting argument, or one shows $G_{2i} \to \Sigma(-1)^j\chi\mathscr{P}^j\alpha_{2i-qj}$ is an inverse. □

In order to prove $C_k \simeq \bigvee_{i \geqslant k} \Sigma^{2i} l$, we use the following result proved in §5. The techniques used in proving this lemma should have uses outside this paper.

LEMMA 3.4. *If X is a locally finite connected spectrum with $H^*(X) \approx \oplus_r \Sigma^{2i_r}A//E_1$, then any A-homomorphism $H^*(l \wedge X) \to H^*(l \wedge l)$ is realized by a map.*

Then 3.4 guarantees existence of a map

$$p\colon \bigvee_{i \geqslant k} \Sigma^{2i} l \wedge l \to l \wedge C_k$$

such that

$$p^*\colon A//E_1 \otimes \bigoplus_{i \geqslant k} \Sigma^{2i}A//E_1 \to \bigoplus_{i \geqslant k} \Sigma^{2i}A//E_1 \otimes A//E_1$$

is the identity homomorphism. Let H be a homotopy inverse to p. Then the composite

$$S^0 \wedge C_k \to l \wedge C_k \xrightarrow{H} \bigvee_{i \geqslant k} \Sigma^{2i} l \wedge l \xrightarrow{\Sigma^{2i} m} \bigvee_{i \geqslant k} \Sigma^{2i} l$$

induces an isomorphism in cohomology, completing the proof of 2.2.

$$[G_{2i} \leftarrow 1 \otimes G_{2i} \leftarrow \Sigma^{2i}1 \otimes 1 \leftarrow \Sigma^{2i}1.]$$

4. Proof of Theorem 2.3. We will need the following result of [**18**].

THEOREM 4.1 [**18**, 5.6]. *If X is a spectrum of finite type and $\{Y_k\}$ is an inverse system of spectra, each of finite type, then there is a spectral sequence converging strongly to $[X, \operatorname{holim}_k \hat{Y}_k]_*$, with $E_2 = \operatorname{Ext}_A(\operatorname{colim}_k H^*Y_k, H^*X)$.*

Let Δ denote the A-module $\operatorname{colim}_k H^*(L_{-k})$, Δ_k the submodule of classes of degree $\geqslant k$, and Δ^{k-1} the quotient Δ/Δ_k. It follows from 4.1 that there is an ASS with

$$E_2^{s,t} \approx \operatorname{Ext}_A^{s,t}(\Delta \otimes A//E_2, \Sigma^{2i-1}A//E_1)$$

converging to $[\Sigma^{2i-1}l, (L \wedge B)_{-\infty}]$. The homomorphism sending $xy^{i+(p-1)j-1} \otimes 1$ to $(-1)^j\Sigma^{2i-1}\mathscr{P}^j\ell$ and $y^{i+(p-1)j}$ to $(-1)^{j+1}\Sigma^{2i-1}\beta\mathscr{P}^j\ell$ gives an element γ_i of $E_2^{0,0}$

which, when restricted to $\mathrm{Ext}_A(\Delta_{2k-1} \otimes A//E_2, \Sigma^{2i-1}A//E_1)$ for any k, is the cohomology homomorphism induced by the restriction to $\Sigma^{2i-1}l$ of the map g: $C_k \to \Sigma L_{2k-1} \wedge B$ of 2.2. [To see that this is A-linear, one verifies that the analogous morphism $\Delta \to \Sigma^{2i-1}A//E_1$ is E_2-linear, extends to $A \otimes_{E_2} \Delta \to \Sigma^{2i-1}A//E_1$, and uses the A-isomorphism $A//E_2 \otimes \Delta \to A \otimes_{E_2} \Delta$.] Because

$$\mathrm{Ext}_A^{s,t}\left(\Delta^{2k-2} \otimes A//E_2, \Sigma^{2i-1}A//E_1\right) \approx \mathrm{Ext}_{E_2}^{s,t}\left(\Delta^{2k-2}, \Sigma^{2i-1}A//E_1\right) = 0$$

if $t - s \geqslant 2k - 2i + 2s(p^2 - 1)$, the restriction

$$\rho_{i,k}^{s,t}\colon \mathrm{Ext}_A^{s,t}\left(\Delta \otimes A//E_2, \Sigma^{2i-1}A//E_1\right) \to \mathrm{Ext}_A^{s,t}\left(\Delta_{2k-1} \otimes A//E_2, \Sigma^{2i-1}A//E_1\right)$$

is injective in the same range and an isomorphism if

$$t \geqslant 2k - 2i + \left(2p^2 - 1\right)(s + 1).$$

Now suppose $d_r(\gamma_i) \neq 0$ in the ASS converging to $[\Sigma^{2i-1}l, (L \wedge B)_{-\infty}]$. Choose $k < i - (p^2 - 1)r$. Then $\rho_{i,k}^{r,r-1}$ is injective, and $\rho_{i,k}^{s,s}$ is surjective for $s \leqslant r$. Hence $d_r(\rho_{i,k}^{0,0}(\gamma_i)) \neq 0$ in the ASS converging to $[\Sigma^{2i-1}l, L_{-2k-1} \wedge B]$, contradicting the assertion of the preceding paragraph that $\rho_{i,k}^{0,0}(\gamma_i)$ is the cohomology homomorphism induced by a map, and hence giving a map $\Sigma^{2i-1}l \to (L \wedge B)_{-\infty}$ whose cohomology effect is as required in 2.3(i).

The maps of 2.3(i) give a map f_1: $\bigvee_{i \in \mathbf{Z}} \Sigma^{2i-1}l \to (L \wedge B)_{-\infty}$. Let ρ_k denote the compatible maps $(L \wedge B)_{-\infty} \to L_{2k-1} \wedge B$. Since each $\pi_j(L_{2k-1} \wedge B)$ is a finite p-group, each $L_{-2k-1} \wedge B$ is p-complete. Thus there are compatible maps

$$(\rho_k f_1)^{\wedge}\colon \left(\bigvee_{i \in \mathbf{Z}} \Sigma^{2i-1}l\right)^{\wedge} \to L_{2k-1} \wedge B,$$

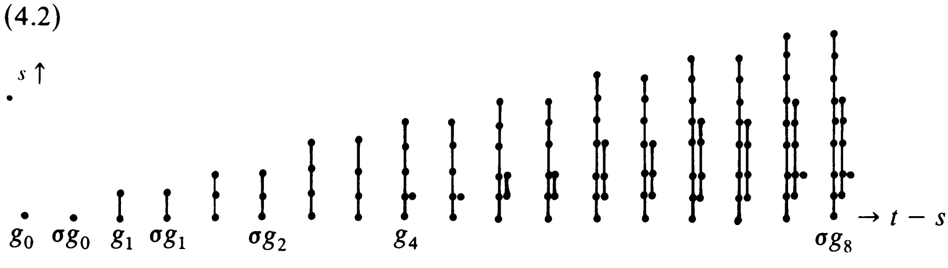
and hence a map f: $(\bigvee_{i \in \mathbf{Z}} \Sigma^{2i-1}l)^{\wedge} \to (L \wedge B)_{-\infty}$.

The E_2-term of the ASS converging to $\pi_*(L_{2k-1} \wedge B)$ is easily calculated (see (4.2)) by minimal resolution. The result is

$$F_p[q_2] \otimes F_p[\sigma]/\sigma^{p-1} \otimes F_p[q_0](g_i\colon i \geqslant 0)/\left(q_0^{i+1}g_i\right),$$

where σ, q_0, q_2, and g_i have bidegrees (s, t) equal to (2,0), (1,1), $(1, 2p^2 - 1)$, and $(0, 2k - 1 + qi)$, respectively. We illustrate for $p = 3$, where dots indicate F_p, and vertical segments multiplication by q_0.

(4.2)



Since all nonzero elements are in even $t - s$, and differentials decrease $t - s$ by 1, all differentials must be zero, and so this chart also presents $\pi_*(L_{2k-1} \wedge B)$, with vertical segments corresponding to multiplication by p.

Calculation of induced homomorphisms of minimal resolutions as in 4.4 shows that the Ext homomorphism induced by a map $f\colon \Sigma^{2k-1+qm+2r}l \to L_{2k-1} \wedge B$ with $0 \leqslant r < p-1$ and cohomology effect as in 2.3(i) is

$$(4.3) \qquad f_*\left(\Sigma^{2k-1+qm+2r}q_0^j q_1^n\right) = \begin{cases} \sigma^r q_0^j q_2^n g_{m-pn} & \text{if } m \geqslant pn, \\ 0 & \text{otherwise.} \end{cases}$$

Let G_m denote the abelian group with generators β_i for $0 \leqslant i < m$ and relations $p^{(m-i)(p+1)}\beta_i$. Then the above ASS calculation shows that

$$G_m \approx \pi_{2j-1}\left(L_{2j-1-q(m(p+1)-1)} \wedge B\right),$$

where β_i corresponds to $q_2^i g_{(m-i)(p+1)-1}$. Hence the inverse system

$$\cdots \to G_{m+1} \to G_m \to \cdots \to G_1 \qquad (\beta_i \mapsto \beta_i)$$

is cofinal in $\{\pi_{2j-1}(L_{-2k-1} \wedge B)\colon k \to \infty\}$ for any j. Hence

$$\pi_{2j-1}\left((L \wedge B)_{-\infty}\right) \approx \lim_k \pi_{2j-1}\left(L_{2k-1} \wedge B\right) \approx \lim_m G_m$$

is the direct product of copies of the p-adic integers with generators β_i. The first isomorphism uses that the derived functor $\mathrm{Rlim}_k \pi_*(L_{-2k-1} \wedge B)$ is 0 because each $\pi_j(L_{-2k-1} \wedge B)$ is finite.

$\pi_{2j-1}((\bigvee \Sigma^{2i-1}l)^\wedge)$ is a direct product of copies of the p-adic integers with generators α_i, which by (4.3) map to β_i (plus possibly elements of higher filtration) under the homomorphism

$$\pi_{2j-1}\left(\left(\bigvee \Sigma^{2i-1}l\right)^\wedge\right) \to \pi_{2j-1}\left(L_{-2k-1} \wedge B\right) = G_m$$

(for appropriate k). These induce an isomorphism into

$$\lim_m G_m \approx \pi_{2j-1}\left((L \wedge B)_{-\infty}\right).$$

PROOF OF 2.3(iii). There are projection maps $(\bigvee \Sigma^{2i-1}l)^\wedge \to \Sigma^{2k-1\wedge}l$ whose product $(\bigvee \Sigma^{2i-1}l)^\wedge \to \prod_{k \in \mathbf{Z}} \Sigma^{2k-1}\hat{l}$ induces an isomorphism in $\pi_*(\ )$.

PROOF OF 2.3(iv). The first equivalence follows exactly as in part (iii) above. To prove the second, we note that the composite

$$\left(\bigvee \Sigma^{qi-1}l\right)^\wedge \to \left(\bigvee \Sigma^{2i-1}l\right)^\wedge \to (L \wedge B)_{-\infty} \to (P \wedge B)_{-\infty}$$

induces an isomorphism in $\pi_*(\ )$ by a calculation similar to 2.3(ii), using that $L_{-qk} \to P_{-qk}$ induces an injection in $H^*(\ )$.

4.4. *Minimal resolutions*. In the proof of 2.3(ii), several statements were made about minimal resolutions. These will now be verified. It suffices to consider the case $k = 0$ of $L_{2k-1} \wedge B$.

From the cohomology description given in the proof of 3.2, $H^*(\Sigma L_{-1} \wedge B)$ splits over A into a direct sum of $p-1$ isomorphic (up to grading) A-modules. The bottom summand S has generators z_i, $i \geqslant 0$, with $|z_i| = qi$, and relations $Q_0(z_i) = Q_1(z_{i-1}) = Q_2(z_{i-p-1})$, where terms with negative subscripts are ignored. We shall show that a homomorphism $f\colon S \to \Sigma^{qm}A//E_1$ sending $z_{m+j} \mapsto (-1)^j \mathscr{P}^j$ for $j \geqslant 0$ induces

$$f^*\colon \mathrm{Ext}_A\left(\Sigma^{qm}A//E_1, F_p\right) \to \mathrm{Ext}_A\left(S, F_p\right)$$

as in the case $r = 0$ of (4.3).

A minimal resolution $(\oplus C_i,\ d)$ of $A//E_1$ is $A \otimes F_p[u_0, u_1]$, where $u_0^i u_1^n$ has degree $i + (q+1)n$ in C_{i+n} and $d(u_0^i u_1^n) = Q_0 \otimes u_0^{i-1} u_1^n + Q_1 \otimes u_0^i u_1^{n-1}$ (terms are ignored if an exponent is negative). A minimal resolution of S is $A \otimes F_p[w_1, w_2, z]$, where $w_1^i w_2^j z^k$ has degree $(q+1)i + (qp+q+1)j + qk$ in C_{i+j} and

$$d\left(w_1^i w_2^j z^k\right) = -Q_1 \otimes w_1^{i-1} w_2^j z^k + Q_0 \otimes w_1^{i-1} w_2^j z^{k+1}$$
$$-Q_2 \otimes w_1^i w_2^{j-1} z^k + Q_1 \otimes w_1^i w_2^{j-1} z^{k+p}.$$

One verifies that the homomorphism f is covered by the homomorphism of minimal resolutions

$$w_1^i w_2^j z^{m-i-pj+k} \to (-1)^k \mathscr{P}^k \otimes u_0^i u_1^j. \tag{4.5}$$

We dualize, naming the dual to $u_0^i u_1^j$ as $q_0^i q_1^j$ and the dual to $w_1^i w_2^j z^k$ as $q_0^i q_2^j g_{k+i}$. This naming is consistent with the Yoneda action of q_0 on these Ext groups. The dual of (4.5) is (4.3).

5. Proof of Lemma 3.4. We will use the following result.

THEOREM 5.1 **[13, 11]**. *There are finite spectra $K(m)$ such that*

$$l \wedge l \simeq \bigvee_{m \geqslant 0} \Sigma^{qm} K(m) \wedge l.$$

As an E_1-module $H^*K(m) \approx L(\nu(m!)) \oplus F$, where F is free, $\nu(\ )$ is the exponent of p in the prime factorization, and $L(m)$ is the E_1-module with generators g_i, $0 \leqslant i \leqslant m$, $|g_i| = qi$, with relations $Q_1 G_i = Q_0 G_{i+1}$ for $0 \leqslant i < m$, $Q_0 G_0$, and $Q_1 G_m$.

Let $L(-m)$ be the E_1-module dual to $L(m)$. Then if $m > 0$, $L(-m)$ has generators h_i, $-m \leqslant i < 0$, $|h_i| = -qi - 1$, with relations $Q_0 h_i = Q_1 h_{i-1}$ for $-m < i < 0$. Most of our work will be directed toward proving the following result. Here the elsewhere we use the change-of-rings theorem without comment.

THEOREM 5.2. *Let X be as in* 3.4. *Let Z be a finite spectrum with $H^*(Z) \approx L(-m) \oplus F$ as E_1-modules, with F free. Then the ASS*

$$\operatorname{Ext}_{E_1}\left(H^*(X \wedge Z),\ F_p\right) \Rightarrow \pi_*(l \wedge X \wedge Z) = l_*(X \wedge Z)$$

collapses.

For any spectra X and Y, let $[l \wedge X, l \wedge Y]_l$ denote the set of l-module maps, using $m\colon l \wedge l \to l$.

PROPOSITION 5.3. $[l \wedge X,\ l \wedge l \wedge Y]_l \approx [X, l \wedge Y]$.

PROOF. If $i\colon S^0 \hookrightarrow l$, then $f \mapsto (m \wedge Y) \circ f \circ (i \wedge X)$ and $l \wedge F \leftarrow F$ are inverse. □

PROOF OF 3.4. Let $W_{m,n} = \Sigma^{q(m+n)} K(m) \wedge K(n)$, $DW_{m,n}$ its dual and $W = \bigvee_{m,n} W_{m,n}$. Let $j\colon l \wedge W \to l \wedge l \wedge l$ be an equivalence given by 5.1. Let X be as in

3.4. There is a commutative diagram

$$\begin{array}{ccc}
[l \wedge l, l \wedge X] & \to & \operatorname{Hom}_A(H^*(l \wedge X), H^*(l \wedge l)) \\
\| & & \| \\
[l \wedge l \wedge l, l \wedge l \wedge X]_l & & \operatorname{Hom}_{E_1}(H^*X, H^*(l \wedge l)) \\
\| & & \| \\
[l \wedge W, l \wedge l \wedge X]_l & & \operatorname{Hom}_{E_1}(H^*X, H^*W) \\
\| & & \| \\
[W, l \wedge X] & \to & \operatorname{Hom}_A(H^*(l \wedge X), H^*W) \\
\| & & \| \\
\prod [W_{m,n}, l \wedge X] & \to & \prod \operatorname{Hom}_A(H^*(l \wedge X), H^*W_{m,n}) \\
\| & & \| \\
\prod \left[S^0, l \wedge X \wedge DW_{m,n}\right] & \to & \prod \operatorname{Hom}_A\left(H^*(l \wedge X \wedge DW_{m,n}), F_p\right)
\end{array}$$

By Lemma 5.10, $DW_{m,n}$ satisfies the hypothesis of Z in 5.2. Thus 3.4 follows from 5.2 and the above diagram. □

Theorem 5.2 will be proved by comparing the ASS with the following spectral sequence of Robinson.

THEOREM 5.4 **[17]**. *If X and Z are spectra of finite type, there is a Kunneth spectral sequence (KSS)*

$$E^2_{s,t} = \operatorname{Tor}^{l_*}_{s,t}(l_*X, l_*Z) \Rightarrow l_*(X \wedge Z).$$

$E^2_{s,t} = 0$ *if* $s > 2$. *There is a single possible differential* d^2: $E^2_{2,t} \to E^2_{0,t+1}$.

We begin by calculating $l_*(X)$ and $l_*(Z)$. Henceforth $\operatorname{Ext}(M)$ is short for $\operatorname{Ext}_{E_1}(M, F_p)$. The next result, 5.6, is well known **[1, 11, 14, 2]**.

DEFINITION 5.5. Let q_0 and q_1 be the canonical generators of bideg$(s,t) = (1,1)$ and $(1, 2p-1)$, respectively, in $\operatorname{Ext}(F_p)$. Let $M(s)$ denote the $F_p[q_0, q_1]$-module with:

if $s \geqslant 0$, generators a_i, $0 \leqslant i \leqslant s$, of bideg$(0, qi)$, and relations $q_1 a_i = q_0 a_{i+1}$, $0 \leqslant i < s$;

if $s < 0$, generators b_{-i}, $0 < i \leqslant s$, of bideg$(0, -iq-1)$, and a of bideg(s,s), and relations $q_1 b_{-i} = q_0 b_{-(i-1)}$, $1 < i \leqslant s$, $q_0 b_{-s}$, and $q_1 b_{-1}$.

THEOREM 5.6. $\operatorname{Ext}(L(s)) \approx M(s)$.

Recall $\pi_*(l) \approx \mathbf{Z}_{(p)}[v]$ with $v\ (= v_1) \in \pi_q(l)$.

DEFINITION 5.7. $N(s)$ is the $\mathbf{Z}_{(p)}[v]$-module with:

if $s \geqslant 0$, generators a_i, $0 \leqslant i \leqslant s$, of degree qi, and relations $va_i = pa_{i+1}$, $0 \leqslant i < s$;

if $s < 0$, generators b_{-i}, $0 < i \leqslant s$, of degree $-iq-1$, and a of degree 0, relations $vb_{-i} = pb_{-(i-1)}$, $1 < i \leqslant s$, pb_{-s}, and vb_{-1}.

THEOREM 5.8. *Let Y be a locally finite connected spectrum, V a finite graded F_p-vector space, and V^* its dual vector space.*

(i) *If H^*Y is E_1-isomorphic to $\oplus_r \Sigma^{2s_r} L(m_r) \oplus (V \otimes E_1)$ with $m_r \geqslant 0$, then*

$$l_*Y \approx \bigoplus \Sigma^{2s_r} N(m_r) \oplus V^*.$$

(ii) *If* $H^*Y \approx \Sigma^s L(-n) \oplus (V \otimes E_1)$, *then* $l_*Y \approx \Sigma^s N(-n) \oplus V^*$.

PROOF. A Bockstein spectral sequence argument as in [**11**, §9D or **9**, 4.17] shows that the free E_1-submodule of H^*Y yields a trivial $\mathbf{Z}_{(p)}[v]$-submodule of l_*Y. Disregarding these, the ASS for l_*Y has E_2-term given from 5.6 and has no possible nonzero differentials. Thus $\oplus \Sigma^{2s_r} M(m_r)$ is an associated graded for l_*Y as a $\mathbf{Z}_{(p)}$-module. In order to choose generators of l_*Y which satisfy the relations of $\oplus \Sigma^{2s_r} N(m_r)$, we use the easily proved fact that in l_*Y all elements of filtration > 1 are in $(p, v) \cdot \{$classes of positive filtration$\}$. We choose generators in order of increasing degree. Choose a generator a_i correct in the associated graded. It will satisfy $\operatorname{filtr}(pa_i - va_{i-1}) > 1$. Then a_i and a_{i-1} can be varied by elements of positive filtration so that the new elements satisfy $pa_i' - va_{i-1}' = 0$. The change in a_{i-2} may require changes in $a_{i-1}, \ldots, a_0$. Since each $N(m_r)$ has only a finite number of a_i's, there is no convergence problem in this procedure. □

Included in the proof of 5.1 in [**11**] is the E_1-splitting

$$A//E_1 \approx \bigoplus \Sigma^{qm} L(\nu(m!)) \oplus F.$$

Thus if X is as in 3.4, there is an E_1-isomorphism

$$H^*X \approx F \oplus \bigoplus_r \Sigma^{2s_r} L(m_r) \quad \text{with } m_r \geqslant 0 \text{ and } F \text{ free.} \tag{5.9}$$

We will need the following lemma.

LEMMA 5.10. *If* $m, n > 0$, *there is an isomorphism of* E_1-*modules*

$$L(m) \otimes L(-n) \approx L(m-n) \oplus (V \otimes E_1),$$

where V *is a graded* F_p-*vector space, with generating function for its dimensions*

$$\frac{x^{-qn-1}(1 - x^{q(m+1-e)})(1 - x^{q(n+e)})}{(1 - x^q)(1 - x^q)},$$

where

$$e = \begin{cases} 0 & \text{if } n < m, \\ 1 & \text{if } n \geqslant m. \end{cases}$$

PROOF. The classification of E_1-modules in [**1** or **2**] and consideration of Q_0- and Q_1-homology implies the general form. Let $p_s(x) = 1 + (1 + x)(x^q + \cdots + x^{sq})$. Then the generating function $g_V(x)$ for V satisfies

$$(1 + x)(1 + x^{q+1}) g_V(x) = p_m(x) \cdot p_n(x^{-1}) - p_{|m-n|}(x^{(-1)^e}).$$

The calculation is then routine. □

Now let X and Z be as in 5.2. Since 5.8 implies that each summand in the E_1-splitting of H^*X and H^*Z gives rise to corresponding summands in l_*X and l_*Z, we can calculate the E_2-terms of 5.2 and 5.4 summand-by-summand. These results are summarized in 5.11. The Ext calculation of 5.2 is performed using 5.6 and 5.10, while the Tor calculation of 5.4 will be presented later. If M is an E_1-module, let $\underline{M}$ denote the underlying graded F_p-vector space. In the Tor calculation of 5.4, this grading refers to $s + t$.

THEOREM 5.11.

H^*X-*summand*	H^*Z-*summand*	*contribution to ASS* E_2 *of* 5.2	*contribution to KSS* E^2 *of* 5.4
E_1	E_1	$\underline{E_1}$ *in* $s=0$	$\underline{E_1}$, *split among* $s=0,1$, *and* 2,
$L(m)$	E_1	$\underline{L(m)}$ *in* $s=0$	$\underline{L(m)}$, *split between* $s=0$ *and* 1,
E_1	$L(-n)$	$\underline{L(-n)}$ *in* $s=0$	$\underline{L(-n)}$, *split between* $s=0$ & 1, $\oplus F_p$ *in* $(s,t)=(2,-1)$ & $(0,0)$,
$L(m)$	$L(-n)$	$M(m-n)\oplus \underline{V}$, *as in* 5.10, *in* $s=0$	$N(m-n)$, *split between* $s=0$ 1, $\oplus\underline{V}$, *as in* 5.10, *in* $s=0$

See 5.12 for a more precise statement of the $N(m-n)$-case.

PROOF OF 5.2. As in the proof of 5.8, [**9**, 4.17] implies that the filtration-0 F_p's cannot support a nonzero differential. Aside from these, E_2 of the ASS of 5.2 consists of infinite q_0-towers in even degrees and finite q_0-towers in odd degrees. [This uses 5.6 and 5.10; an example is given below.] The only possible differentials might go from infinite towers to (the top part of) finite towers. By 5.11 and 5.12, $E^2_{0,*}$ in the KSS of 5.4 contains finite cyclic summands isomorphic to the summands in $l_*(X\wedge Z)$ which would be caused by these finite towers in the ASS if they are not hit by a differential. If the finite towers are hit by a differential in the ASS, the corresponding summands must also be hit by a differential in the KSS, since the E_∞-terms of the two spectral sequences must be associated gradeds of the same graded abelian group. This differential can only come from $E^2_{2,*}$, where there are only F_p's.

By 5.11, the F_p's in the KSS correspond exactly (in number and degree) to those in the ASS, except for pairs (in the third case of 5.11) which have the possibility of self-annihilation. (And, indeed, this self-annihilation must occur.) If d_2 on an F_p in the KSS were to hit a nonzero element in a summand of order greater than p, this would eliminate the possibility of agreement of the F_p's in E_∞ of the spectral sequences, which must be present.

We illustrate with $H^*Z=L(-4)$ and

$$H^*(X)=L(6)\oplus\Sigma^qE_1\oplus\Sigma^{3q}L(1).$$

Of course, H^*X as in 3.4 will have infinitely many summands, but the simple case here contains all pertinent features. $\mathrm{Ext}^{s,t}_{E_1}(H^*(X\wedge Z),F_p)$ is given by Figure 1

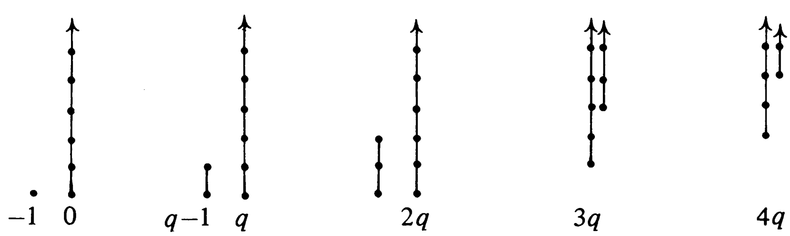


FIGURE 1

TABLE 1

number of additional F_p's in $s = 0$, $t = ---$

m	$mq - 1$ in ASS	mq in ASS	$mq - 1$ in KSS
-4	1		1
-3	3	1	2
-2	4	1	3
-1	6	1	5
0	6	1	5
1	5	1	5
2	5		5
3	5		5
4	3		3
5	2		2
6	1		1

(where dots are F_p's and vertical segments multiplication by q_0) plus filtration zero F_p's as given in Table 1. $\mathrm{Tor}_{s,t}(l_*X, l_*Z)$ contains the F_p's in $s = 0$ in the last column above, plus Figure 2, where a number e means $\mathbf{Z}/p^e$ ($e = \infty$ means $\mathbf{Z}$). The only possible differential in the KSS is on the lone class in $s = 2$. It must be nonzero in order to give agreement in $l_{q+1}(X \wedge Z)$. Then there can be no nonzero differentials in the ASS in order to give agreement. □

All that remains is the verification of the last column of 5.11. Tor( ) will always mean $\mathrm{Tor}^{l_*}(\ ) = \mathrm{Tor}^{\mathbf{Z}_{(p)}[v]}(\ )$. The first three cases reduce to calculating the homology of

$$\begin{array}{ccccccccc}
 & & C_2 & & C_1 & & C_0 & & \\
 & & \| & & \| & & \| & & \\
0 & \to & \Sigma^q M & \to & M \oplus \Sigma^q M & \to & M & \to & 0 \\
 & & \Sigma^q z & \to & vz - p\Sigma^q z & & & & \\
 & & & & x + \Sigma^q y & \to & px + vy & &
\end{array}$$

for $M = F_p$, $N(m)$, and $N(-n)$, since for $s = 0, 1, 2$, $\mathrm{Tor}_s(M, F_p) \approx \mathrm{Tor}_s(F_p, M)$ is the sth homology group of this sequence. These are readily verified to be F_p-vector spaces with bases as in Table 2. To visualize these calculations, the reader may be aided by charts of $N(m)$ and $N(-n)$ in [2]. The total degrees $s + t$ of these elements are as claimed in the first three rows of the last column of 5.11.

				1				$s = 2$
1	1	1	6, 1	5, 1	4	3, 1	2	$s = 1$
1	1	1	1, 1 ∞	2 ∞, 1	3 ∞	∞, ∞	∞, ∞	$s = 0$
$-3q$	$-2q$	$-q$	0	q	$2q$	$3q$	$4q$	$s + t$

FIGURE 2

TABLE 2

M	$s = 0$	$s = 1$	$s = 2$
F_p	a	$a,\ \Sigma^q a$	$\Sigma^q a$
$N(m)$	$a_0, \dots, a_m$	$a_i - \Sigma^q a_{i-1}$: $0 < i \leqslant m$	$\varnothing$
$N(-n)$	$b_{-n}, \dots, b_{-1}, a$	$b_{-n}, b_i - \Sigma^q b_{i-1}$: $-n < i < 0,\ \Sigma^q b_{-1}$	$p^{n-1}\Sigma^q b_{-1}$

Let $T(-s)$ denote the submodule of $N(-s)$ generated by the b_i's. Thus $N(-s) \approx T(-s) \oplus N(0)$. Let I denote the ideal (p, v) in $\mathbf{Z}_{(p)}[v]$. Note that the submodule $I^k \cdot N(m)$ is isomorphic to $N(m+k)$; in the sequel, $N(m)/N(m+k)$ means $N(m)/I^k \cdot N(m)$.

THEOREM 5.12. *$\operatorname{Tor}_s(N(m), N(-n)) = 0$ if $s > 1$.*

$$\operatorname{Tor}_0(N(m), N(-n)) \approx \underline{V} \oplus N(m)$$

$$\oplus \begin{cases} T(-(n-m)) & \text{if } n > m, \\ 0 & \text{if } n \leqslant m, \end{cases} \qquad \text{with } \underline{V} \text{ as in } 5.10.$$

$$\operatorname{Tor}_1(N(m), N(-n)) \approx \begin{cases} \Sigma^{-1}N(0)/N(m) & \text{if } n \geqslant m, \\ \Sigma^{-1}N(m-n)/N(m) & \text{if } m > n. \end{cases}$$

Thus there is a short exact sequence of graded abelian groups (where grading in $\operatorname{Tor}_{s,t}$ is $s+t$).

$$0 \to \operatorname{Tor}_{0,*}(N(m), N(-n)) \to \underline{V} \oplus N(m-n) \to \operatorname{Tor}_{1,*}(N(m), N(-n)) \to 0.$$

This is the precise version of the last case of 5.11.

PROOF.

$$\operatorname{Tor}_s(N(m), N(-n)) \approx \operatorname{Tor}_s(N(m), T(-n)) \oplus \operatorname{Tor}_s(N(m), N(0)),$$

and the latter summand is $N(m)$ if $s = 0$, and 0 otherwise. $\operatorname{Tor}_0(N(m), T(-n)) \approx N(m) \otimes T(-n)$; all $\otimes$ ing is over $\mathbf{Z}_{(p)}[v]$. This has an F_p-subspace with basis

$$\mathbf{B} = \{a_i b_{-j} - a_{i+1}b_{-j-1}: 0 \leqslant i < m, 1 \leqslant j \leqslant n\}$$
$$\cup \{a_i b_{-1}: \max(0, m-n) < i \leqslant m\},$$

where $\otimes$-signs are omitted, and b_{-n-1} is to be interpreted as 0. For example, $p(a_i b_{-j} - a_{i+1}b_{-j-1}) = va_i b_{-j-1} - va_i b_{-j-1}$. This basis is also annihilated by v.

The generating function for this subspace agrees with that of V of 5.10. If $m \geqslant n$, the $mn + n$ elements in $\mathbf{B}$ span $N(m) \otimes T(-n)$, while if $n > m$, $\mathbf{B}' = \mathbf{B} \cup \{a_m b_{-j}: m < j \leqslant n\}$ spans $N(m) \otimes T(-n)$. These elements $a_m b_{-j}$ have the same degrees and satisfy the same relations as the generators of $T(-(n-m))$. This defines a homomorphism

$$\underline{V} \oplus T(-(n-m)) \to N(m) \otimes T(-n).$$

An inverse ϕ is easily obtained by sending $a_i b_{-j}$ to the appropriate sum of elements of $\mathbf{B}'$; one easily verifies that

$$\phi(pa_i b_{-j}) = \phi(va_{i-1}b_{-j}) = \phi(pa_{i-1}b_{-j+1}).$$

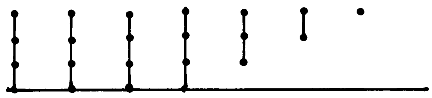


FIGURE 3

There is an exact sequence
(5.13)

$$0 \to \mathrm{Tor}_1(N(m), T(-n)) \to \bigoplus_{i=1}^{m} \Sigma^{qi}T(-n)$$

$$\xrightarrow{d} \bigoplus_{i=0}^{m} \Sigma^{qi}T(-n) \to N(m) \otimes T(-n) \to 0, \qquad d(\sigma_i b_{-j}) = p\sigma_i b_{-j} - v\sigma_{i-1}b_{-j},$$

where $\sigma_i b$ is the generator b in $\Sigma^{qi}T(-n)$.

For $m > n$, let $z_j = \Sigma_{i=1}^{n} \sigma_{j+i} b_{-i}$ for $0 \leqslant j \leqslant m-n$. Then $z_j \in \ker(d)$, $\mathrm{order}(z_j) = \mathrm{order}(b_{-1}) = p^n$, and $vz_j = pz_{j+1}$. Thus $\{z_0, \ldots, z_{m-n}\}$ generate a submodule of $\ker(d)$ isomorphic to $\Sigma^{-1}N(m-n)/N(m)$. For example, a chart representing this graded abelian group when $m = 7$ and $n = 4$ is given below. Towers are q units apart; bottoms of towers represent z_0, z_1, z_2, z_3, vz_3, v^2z_3, and v^3z_3, and vertical segments are $\cdot p$. See Figure 3. This can be shown to be all of $\ker(d)$ by calculating orders of groups in (5.13).

Similarly, if $m \leqslant n$, then $p^{n-m}\Sigma_{i=1}^{m}\sigma_i b_{-i}$ is in $\ker(d)$, has order p^m, and generates a submodule isomorphic to $\Sigma^{-1}N(0)/N(m)$, which must be all of $\ker(d)$ by a counting argument. □

6. The spectrum $(P \wedge \mathrm{BP})_{-\infty}$. In this section we prove Theorem 1.6 by proving the analogue of Theorem 2.3 with all l's and B's replaced by BP. We call this result 6.3. Let E denote the exterior subalgebra of A generated by all Milnor primitives Q_i, $i \geqslant 0$. Then $H^*\mathrm{BP} = A//E$. By 4.1 and the change-of-rings theorem there is an ASS with $E_2 = \mathrm{Ext}_E(\Delta, \Sigma^{2i-1}A//E)$ converging strongly to $[\Sigma^{2i-1}\mathrm{BP}, (L \wedge \mathrm{BP})_{-\infty}]$. Since $A//E$ is 0 in odd degrees and all Q_i have odd degree, $A//E$ is a trivial E-module, and hence the E_2-term is $\prod \mathrm{Ext}_E(\Delta, \Sigma^m F_p)$, where the product is over odd integers m (oft repeated) $\geqslant 2i-1$. But $\mathrm{Ext}_E^{s,t}(\Delta, F_p)$ is 0 if $t-s$ is even (see below), and so the ASS is concentrated in even values of $t-s$ and hence collapses. Thus the homomorphism $\Delta \otimes A//E \to \Sigma^{2i-1}A//E$ sending $xy^{i-1+(p-1)j} \otimes 1$ to $(-1)^j\mathcal{P}^j\ell$ is realized by a map as in 6.3(i).

As in the proof of 2.3(ii), the maps constructed above induce a map as in 6.3(ii). The minimal resolution argument 4.4 extends without difficulty to show that

$$\mathrm{Ext}_A(A//E, F_p) \approx F_p[q_0, q_1, \ldots],$$

$$\mathrm{Ext}_A(\Delta_{2k-1} \otimes A//E, F_p) \approx F_p[q_2, q_3, \ldots] \otimes F_p[\sigma]/\sigma^{p-1}$$
$$\otimes F_p[q_0](g_i\colon i \geqslant 0)/(q_0^{i+1}g_i),$$

and that for $0 \leqslant r < p-1$ a map $f\colon \Sigma^{2k-1+qm+2r}\mathrm{BP} \to L_{2k-1} \wedge \mathrm{BP}$ with cohomology effect as in 6.3(i) induces Ext homomorphism

(6.1) $$f_*(\sigma^{2k-1+qm+2r}q_0^j q_1^{n_1} \cdots q_k^{n_k})$$
$$= \begin{cases} \sigma^r q_0^j q_2^{n_1} \cdots q_{k+1}^{n_k} g_{m-pn_1-\cdots-p^k n_k} & \text{if } m \geqslant pn_1 + \cdots + p^k n_k, \\ 0 & \text{otherwise.} \end{cases}$$

For any integer j, the homomorphism

$$\pi_{2j-1}\left(\left(\bigvee \Sigma^{2i-1}\mathrm{BP}\right)^{\wedge}\right) \to \pi_{2j-1}((L \wedge \mathrm{BP})_{-\infty})$$

is an isomorphism of direct products of copies of the p-adic integers indexed by finite sequences $(n_1, \ldots, n_k)$ of nonnegative integers, corresponding to $q_1^{n_1} \cdots q_k^{n_k}$ (suspended appropriately) and $q_2^{n_1} \cdots q_{k+1}^{n_k}$ (on an appropriate g), respectively. By (6.1), the isomorphism is (under the identifications of the previous sentence) the identity homomorphism, at least up to elements of higher filtration, establishing 6.3(ii). The proofs of 2.3(iii), (iv) are easily adapted to BP, yielding 6.3(iii), (iv).

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